First passage times of Lévy flights coexisting with subdiffusion

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We investigate both analytically and numerically the first passage time (FPT) problem in one dimension for anomalous diffusion processes in which Lévy flights and subdiffusion coexist. We analyze the FPT for three subclasses of Lévy stable motions: (i) symmetric Lévy motions characterized by Lévy index μ, $0<\mu<2$, and skewness parameter $\beta=0$, (ii) one-sided Lévy motions with μ, $0<\mu<1$, and skewness $\beta=1$, and (iii) two-sided skewed Lévy motions, the extreme case, $1<\mu<2$, and skewness $\beta=-1$. In all three cases the waiting times between successive jumps are heavy tailed with index α. We show that in all three cases the FPT distributions are power laws. Our findings extend earlier studies on FPTs of Lévy flights by considering the interplay between long rests and the Lévy long jumps.

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I. INTRODUCTION

Much attention has been devoted to the fractional Fokker-Planck equations (FFPEs) describing anomalous diffusion under the influence of an external field [1–4]. These equations provide a useful approach for the description of different types of dynamics in complex systems which are governed by anomalous diffusion and nonexponential relaxation patterns [5].

The FFPEs can be derived from the generalized master equation or the continuous-time random walk (CTRW) model as shown in Refs. [6,7]. In the CTRW model it has been usually assumed that a random walker performs jumps of step lengths chosen from a given well behaved probability density function (PDF). In the jump process the walker resides on the different sites visited for times chosen randomly from a finite-mean, or infinite-mean PDF. The former case can be described by the simple diffusion equation whereas in the latter case heavy tailed waiting times (temporal-$t^{-(1+\alpha)}$, $0<\alpha<1$) cause slowly decaying memory effects which give rise to the FFPE. However, the step length PDF can also be heavy tailed ($\sim|x|^{-(1+\mu)}$, $0<\mu<2$), corresponding to Lévy flights, and can lead to a spatial-FFPE [8]. While the temporal-FFPE leads to subdiffusion, the spatial-FFPE is related to superdiffusion; namely, two different anomalies. Of special interest are processes where the temporal heavy tails coexist with Lévy flights.

The interest in Lévy flights and their first passage properties [9–12] has been mainly devoted to symmetric flights which are well described by the spatial-FFPE. Less investigated have been other types of Lévy motions such as one-sided or skewed [11]. In general, Lévy stable PDFs are expressed in terms of their characteristic function $\lambda_{\mu,\beta}(k;\sigma,\tau)$ [12–14]

$$\lambda_{\mu,\beta}(x;\sigma,\tau) = \int_{-\infty}^{-\infty} \frac{dk}{2\pi} e^{-ikx} \lambda_{\mu,\beta}(k;\sigma,\tau), \tag{1}$$

where

$$\lambda_{\mu,\beta}(k;\sigma,\tau) = \exp\left[-\tau|k|^{\mu}\left(1 - i\beta\frac{k}{|k|}\omega(k,\mu)\right) + i\sigma k\right] \tag{2}$$

and

$$\omega(k,\mu) = \begin{cases} \tan\dfrac{\pi\mu}{2} & \text{if } \mu \neq 1, \\ -\dfrac{2}{\pi}\ln|k| & \text{if } \mu = 1. \end{cases} \tag{3}$$

The characteristic function and, respectively, the Lévy stable PDF are determined by the parameters μ, β, σ, and τ. The exponent $\mu\in(0,2]$ is the Lévy index, $\beta\in[-1,1]$ is the skewness parameter, σ is the shift parameter, which is a real number, and $\tau>0$ is a scale parameter. The shift and scale parameters play a lesser role and can be actually eliminated by transformations [11]. In order to explore the coexistence of general Lévy flights and subdiffusion we formulate the problem in terms of subordination of the processes [16–18].

In the present work we study in one dimension the statistical properties of the first passage times (FPT) [9] processes for which temporal and spatial tails coexist. We address the problem of how long it takes a random walker, starting at the origin, to cross or arrive at a fixed target ($=d>0$) or barrier. The indices α (time), μ (space) and the skewness parameter β play the major role in our considerations; α and μ define the asymptotic decay of the PDF, whereas β defines the asymmetry of the distribution.

In what follows we investigate the FPT PDFs which correspond to three different Lévy motions: (i) symmetric μ-stable Lévy motion, $\mu\in[0,2]$, and skewness $\beta=0$ (ii) one-sided μ-stable Lévy motion, $\mu\in[0,1]$, and skewness $\beta=1$ and (iii) two-sided extremal μ-stable Lévy motion, μ

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$\in [1,2]$, and skewness $\beta=-1$. In each case the waiting times between successive jumps are heavy tailed. We show, both analytically and numerically, that the FPT PDFs display power law behaviors. We show that the FPT PDFs are strongly related to the interplay between long rests and long jumps of the system described by the FFPE. The procedure of generating Lévy stable distribution is based on the method of Chambers *et al.* [19] (for other algorithms we refer the readers to Refs. [20,21]).

II. SYMMETRIC CASE

The fractional Fokker-Planck equation (FFPE) describing the coexistence between subdiffusion and Lévy flights under the influence of a constant potential, is given in Ref. [2]:

$$\frac{\partial p(x,t)}{\partial t} = {}_0D_t^{1-\alpha}\nabla^{\mu}p(x,t). \tag{4}$$

Here, the operator

$${}_0D_t^{1-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_0^t (t-s)^{\alpha-1}f(s)ds, \tag{5}$$

$0<\alpha<1$, is the fractional derivative of the Riemann-Liouville type and ∇^{μ}, $0<\mu\leq 2$ is the Riesz fractional derivative with the Fourier transform $\mathcal{F}\{\nabla^{\mu}f(x)\}=-|k|^{\mu}\tilde{f}(k)$ [22]. The occurrence of the operator ${}_0D_t^{1-\alpha}$ in Eq. (4) stems from the heavy tailed waiting times between successive jumps of the particle, whereas ∇^{μ} is related to the heavy tailed distributions of Lévy jumps in the underlying continuous-time random walk (CTRW) scheme. Equation (4) can be derived from a generalized master equation [6]. For $\mu=2$, we obtain the FFPE describing subdiffusion in accordance with the mean-squared displacement [2], while for $\alpha=1$, Eq. (4) reduces to the Markovian Lévy flight [23]. The case $\mu=2$, $\alpha=1$ corresponds to the standard Fokker-Planck equation.

In a recent paper [17], it has been shown that the solution of the FFPE (4) is equal to the PDF of the subordinated process

$$Y(t) = X(S_t). \tag{6}$$

Here the parent process $X(\tau)$ is the symmetric μ-stable Lévy motion [24], with the Fourier transform $(0<\mu<2)$

$$\langle e^{ikX(\tau)}\rangle = e^{-\tau|k|^{\mu}}, \tag{7}$$

corresponding to $0<\mu<2$, $\sigma=0$ and $\beta=0$ in Eq. (2). The subordinator S_t, which is independent of $X(\tau)$, is defined as

$$S_t = \inf\{\tau : U(\tau) > t\}. \tag{8}$$

Here, $U(\tau)$ denotes a strictly increasing α-stable Lévy motion [24]—i.e., an α-stable process with Laplace transform

$$\langle e^{-kU(\tau)}\rangle = \exp\left[-\frac{\tau k^{\alpha}}{\cos(\pi\alpha/2)}\right], \tag{9}$$

where $0<\alpha<1$. The processes $X(\tau)$ and $U(\tau)$ are assumed to be independent. Some interesting physical properties of the inverse-time α-stable subordinator S_t have been discussed in Refs. [25–29]. The role of the subordinator S_t in the stochastic representation (6) is analogous to the role of the fractional Riemann-Liouville derivative (5) in the FFPE (4), since S_t appears in a natural way as the limit process of the CTRW scheme with heavy tailed waiting-time distributions between successive jumps of a particle. The process S_t is responsible for the subdiffusive behavior of the system, whereas the parent process $X(\tau)$ introduces the Lévy flight-type behavior (long jumps of a particle). The subordinated process $X(S_t)$ combines both these characteristics, resulting in competition between subdiffusion and Lévy flights [2,17] [for a typical trajectory see Fig. 1(a)].

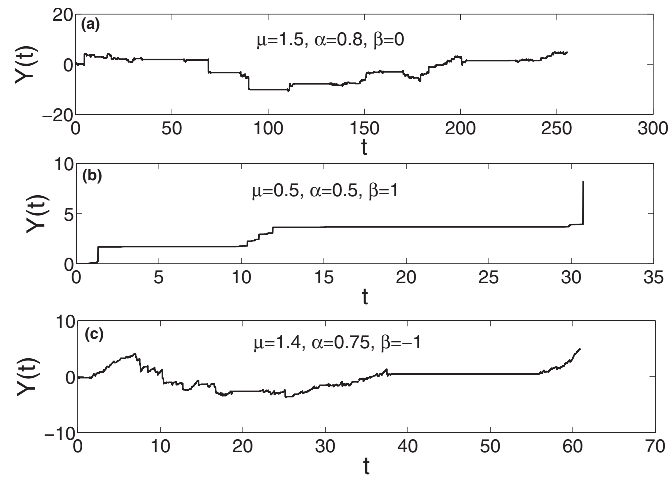

FIG. 1. Typical trajectories of the subordinated anomalous diffusion process $Y(t)=X(S_t)$ defined by the fractional Fokker-Planck equation, describing the competition between subdiffusion (different α values) and Lévy flights (different μ values), under the influence of a constant potential. Three different cases are shown: (a) symmetric Lévy flights $\beta=0$, (b) one-sided Lévy flights $\beta=1$, and (c) two-sided skewed Lévy flights, the extreme case $\beta=-1$. In all three cases $\Delta t=0.01$.

Here we are interested in the computation of the first passage times (FPTs), namely, how long will it take a walker, starting at the origin, to hit (or cross) a barrier positioned at point d? Given a stochastic process $Z(t)$ and a barrier $d>0$, we set

$$\tau_Z(d) = \inf\{t \geq 0 : Z(t) > d\}$$

to be the FPT. In what follows, we investigate the properties of the FPT $\tau_Y=\tau_Y(d)$ corresponding to the anomalous diffusion process $Y(t)=X(S_t)$ defined in Eq. (6). We show that the PDF of τ_Y displays power law behavior.

Let us begin with recalling a fundamental result for FPTs, known as the Sparre-Andersen theorem [30,31]. It states that for any discrete-time random walk process with independent steps chosen from any continuous, symmetric distribution, the FPT PDF decays asymptotically as $n^{-3/2}$. Taking advantage of this result, we immediately obtain that the FPT τ_X corresponding to the symmetric μ-stable Lévy motion $X(\tau)$, displays power law behavior. The PDF $p_{\tau_X}(n)$ of τ_X satisfies

$$p_{\tau_X}(n) \propto n^{-3/2} \tag{10}$$

as $n \to \infty$ and is independent of μ. Knowing the FPT distribution of the process $X(\tau)$, we are able to calculate the distribution of the FPT τ_Y corresponding to the anomalous diffusion $Y(t)=X(S_t)$. Recall that the subordinator is defined as $S_t=\inf\{\tau: U(\tau)>t\}$, therefore the following key relation between τ_X and τ_Y:

$$\tau_Y \overset{d}{=} U(\tau_X) \tag{11}$$

holds. Here, "$\overset{d}{=}$" stands for "equal in law." The above formula is the immediate consequence of the equality

$$S_{\tau_Y} = \tau_X. \tag{12}$$

Note that if we put $U(\cdots)$ on both sides of Eq. (12), we obtain Eq. (11).

Taking advantage of Eq. (11) and using the total probability formula, we get that the PDF $p_{\tau_Y}(y)$ of τ_Y is given by

$$p_{\tau_Y}(y) = \int_0^\infty u(y,\tau) p_{\tau_X}(\tau) d\tau, \tag{13}$$

where $u(y,\tau)$ and $p_{\tau_X}(\tau)$ are the PDFs of $U(\tau)$ and τ_X, respectively. Since the process $U(\tau)$ is $1/\alpha$ self-similar, its PDF satisfies $u(y,\tau)=\frac{1}{\tau^{1/\alpha}}u(\frac{y}{\tau^{1/\alpha}})$, where $u(y)=u(y,1)$. Therefore, after the change of variables $z=\frac{y}{\tau^{1/\alpha}}$, we obtain

$$p_{\tau_Y}(y) = \int_0^\infty \frac{y^{\alpha-1}}{z^\alpha} u(z) p_{\tau_X}\left[\left(\frac{y}{z}\right)^\alpha\right] dz.$$

For fixed $z \in \mathbb{R}_+$, we get from Eq. (10) that the kernel function in the above integral is asymptotically equivalent to $y^{-(\alpha/2+1)} z^{\alpha/2} u(z)$ as $y \to \infty$. Since the function $z^{\alpha/2}u(z)$ is integrable, we finally get from the dominated convergence theorem that the FPT PDF of $Y(t)$ satisfies

$$p_{\tau_Y}(y) \propto y^{-(\alpha/2+1)} \tag{14}$$

as $y \to \infty$. This result, which has been obtained earlier in Refs. [8,9], is in a very good agreement with our numerical results (see Fig. 2) obtained with the help of the method developed in Ref. [17]. Let us note that in the limit case $\alpha \to 1$ we recover the Sparre-Andersen-type behavior of the PDF. For $\alpha<1$ the tail of $p_{\tau_Y}(y)$ decays slower than $y^{-3/2}$, and therefore the PDF in Eq. (14) is broader than in Eq. (10). In both cases the mean of the FPT PDF diverges.

III. ONE-SIDED CASE

In this section, we investigate the FPT problem for another important class of anomalous diffusion processes, with the structure similar to the one introduced in Eq. (6). Let $X(\tau)$ be the one-sided μ-stable Lévy motion, i.e., the strictly increasing μ-stable process $0<\mu<1$ with independent, stationary, and non-negative increments [24]. The Laplace transform of $X(\tau)$ is given by

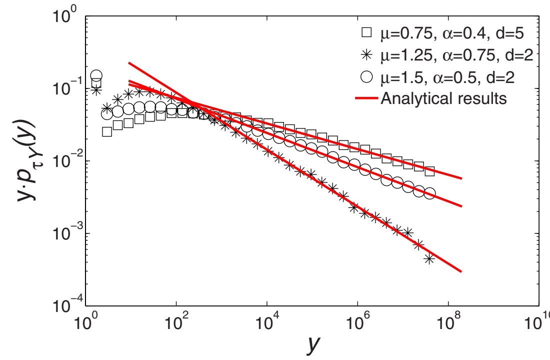

FIG. 2. (Color online) FPT PDFs $p_{\tau_Y}(y)$ of the anomalous diffusion $Y(t)=X(S_t)$, where $X(\tau)$ is the symmetric μ-stable Lévy motion and S_t is the inverse α-stable subordinator, with different values of α, μ, and d. Plotted is $yp_{\tau_Y}(y)$ vs y on a log-log scale. The red (solid) lines correspond to the slope "$-\alpha/2$" [Eq. (14)]. In all cases $\Delta t=0.01$ and 10^6 realizations.

$$\langle e^{-kX(\tau)}\rangle = \exp\left[-\frac{\tau k^\mu}{\cos(\pi\mu/2)}\right], \tag{15}$$

which is equivalent to the characteristic function (2) with $0<\mu<1$, $\beta=1$, and $\sigma=0$ [11]. A corresponding FFPE was proposed in Ref. [32].

Let S_t be defined as in Eq. (8) and independent of $X(\tau)$. In what follows, we examine the properties, in particular the FPT, of the anomalous diffusion process defined as

$$Y(t) = X(S_t). \tag{16}$$

Similarly to the process discussed in the previous section, $Y(t)$ is also obtained via subordination [for a typical trajectory see Fig. 1(b)]. However, the symmetric parent process in representation (6) is replaced here with the one-sided μ-stable Lévy motion $X(\tau)$. The physical interpretation of both components in the subordination is alike: S_t is responsible for the subdiffusive behavior of the system (long rests of the particle) while the parent process $X(\tau)$ introduces the Lévy flight-type behavior (long jumps of the particle). However, the jumps of the parent process $X(\tau)$ here are only positive, and therefore, the paths of $Y(t)$ are nondecreasing. Accordingly, the spatial properties of $Y(t)$ defined in Eq. (16) are significantly different from the properties of the model examined in the foregoing section. Recall that the process introduced in Eq. (6) was symmetric.

Now, we turn to the problem of finding the distribution of the FPT τ_Y corresponding to the anomalous diffusion (16). Repeating the argumentation from the previous section, we get that

$$\tau_Y \overset{d}{=} U(\tau_X), \tag{17}$$

where $U(\tau)$ is characterized by Eq. (9) and $\tau_X=\tau_X(d)$ is the FPT of the strictly increasing μ-stable Lévy motion $X(\tau)$. Using the definition

$$\tau_X(d) = \inf\{\tau \geqslant 0: X(\tau) > d\}$$

and $1/\mu$ self-similarity of $X(\tau)$, we obtain

$$P[\tau_X(d) \leqslant \tau] = P[X(\tau) \geqslant d] = P\{[X(1)/d]^{-\mu} \leqslant \tau\}.$$

Therefore, the distribution of $\tau_X(d)$ is equal to the distribution of the random variable $[d/X(1)]^{\mu}$. Computing the moments of the last random variable shows [33] that its Laplace transform is given by $\langle e^{-k(d/X(1))^{\mu}}\rangle = E_{\mu}(-kd^{\mu})$. Here, the function $E_{\mu}(z)$ is the Mittag-Leffler function

$$E_{\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\mu+1)}.$$

As a consequence, we obtain that the Laplace transform of $\tau_X(d)$ equals

$$\langle e^{-k\tau_X(d)}\rangle = E_{\mu}(-kd^{\mu}).$$

Using the above formula together with Eqs. (9) and (17), we finally obtain that the Laplace transform of $\tau_Y = \tau_Y(d)$ is given by

$$\langle e^{-k\tau_Y(d)}\rangle = E_{\mu}\left(-\frac{k^{\alpha}d^{\mu}}{\cos(\pi\alpha/2)}\right). \tag{18}$$

Now, we will use the Tauberian theorems to establish the tail properties of $\tau_Y(d)$. We have

$$\int_0^{\infty} e^{-ky}P[\tau_Y(d) > y]dy = \frac{1-\langle e^{-k\tau_Y(d)}\rangle}{k} = \frac{1-E_{\mu}\left(-\frac{k^{\alpha}d^{\mu}}{\cos(\pi\alpha/2)}\right)}{k} \sim \frac{d^{\mu}}{\cos(\pi\alpha/2)\Gamma(\mu+1)}k^{\alpha-1}$$

as $k \to 0$. Thus, from the Tauberian theorem (see Ref. [34]), we get that $\tau_Y(d)$ displays power law behavior with

$$P[\tau_Y(d) > y] \sim \frac{y^{-\alpha}d^{\mu}}{\cos(\pi\alpha/2)\Gamma(\mu+1)\Gamma(1-\alpha)}$$

as $y \to \infty$. Further, the PDF of $\tau_Y(d)$ satisfies

$$p_{\tau_Y}(y) \propto y^{-(\alpha+1)} \tag{19}$$

as $y \to \infty$. Here again the PDF exponent is independent of μ. The above formula for the asymptotic behavior of $p_{\tau_Y}(y)$ is in a very good agreement with our numerical results PDF (see Fig. 3). A comparison of Eqs. (14) and (19) indicates that the PDF in the one-sided case is expected to decay faster than the PDF for the symmetric process discussed in Sec. II. This result can be easily justified, since it is enough to notice that the strictly increasing process is expected to cross the barrier faster than the process with both positive and negative jumps.

IV. TWO-SIDED SKEWED CASE

The last anomalous diffusion model we considered is obtained by replacing the symmetric parent process in Eq. (6) with the two-sided skewed μ-stable Lévy motion $X(\tau)$,

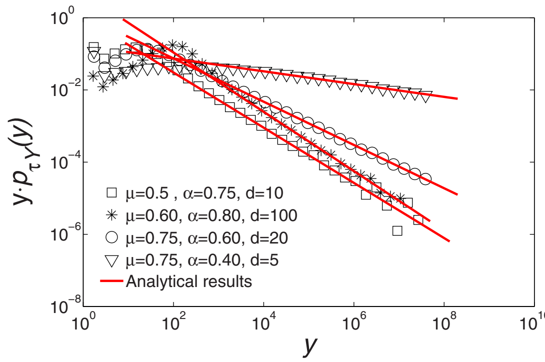

FIG. 3. (Color online) FPT PDFs $p_{\tau_Y}(y)$ of the anomalous diffusion $Y(t)=X(S_t)$, where $X(\tau)$ is the one-sided μ-stable Lévy motion and S_t is the inverse α-stable subordinator, with different values of α, μ, and d. Plotted is $yp_{\tau_Y}(y)$ vs y on a log-log scale. The red (solid) lines correspond to the slope "$-\alpha$" [Eq. (19)]. In all cases $\Delta t=0.01$ and 10^6 realizations.

where $1<\mu<2$ and the skewness parameter $\beta=-1$. The process $X(\tau)$ possesses large jumps in the negative direction and increases continuously. The Fourier transform of $X(\tau)$ is given by

$$\langle e^{ikX(\tau)}\rangle = \exp\left[-\tau|k|^{\mu}\left(1+i\frac{k}{|k|}\tan(\pi\mu/2)\right)\right]$$

with $1<\mu<2$ [see Eq. (2) for $\beta=-1$ and $\sigma=0$]. The FPT problem for the process $X(\tau)$ was studied in Refs. [11,35]. It was shown that the Laplace transform of the FPT $\tau_X(d)$ is equal to

$$\langle e^{-k\tau_X(d)}\rangle = \exp\left[-\frac{\rho^{\mu'}k^{\mu'}}{\cos(\pi\mu'/2)}\right], \tag{20}$$

where $\mu' = 1/\mu$ and

$$\rho = -[d\cos(\pi\mu'/2)]^{\mu}\cos(\pi\mu/2).$$

Now, we turn to the problem of finding the FPT distribution of the anomalous diffusion process $Y(t)=X(S_t)$ with $X(\tau)$ being the above introduced two-sided skewed μ-stable Lévy motion and S_t defined in Eq. (8) [for a typical trajectory see Fig. 1(c)]. A FFPE corresponding to this model was introduced in Ref. [32]. Similarly, as in the two previous sections, we have the following relation between the FPTs τ_X and τ_Y corresponding to $X(\tau)$ and $Y(t)$, respectively:

$$\tau_Y \overset{d}{=} U(\tau_X). \tag{21}$$

Recall that $U(\tau)$ is characterized by Eq. (9). Now, using the above relation together with Eqs. (9) and (20) we are able to calculate the Laplace transform of τ_Y. We have

$$\langle e^{-k\tau_Y(d)}\rangle = \exp\left[-\frac{\rho^{\mu'}k^{\alpha\mu'}}{\cos(\pi\mu'/2)[\cos(\pi\alpha/2)]^{\mu'}}\right]. \tag{22}$$

Thus, we have proved that $\tau_Y(d)$ is a α/μ-stable random variable. As an immediate consequence, we get that the PDF of $\tau_Y(d)$ satisfies

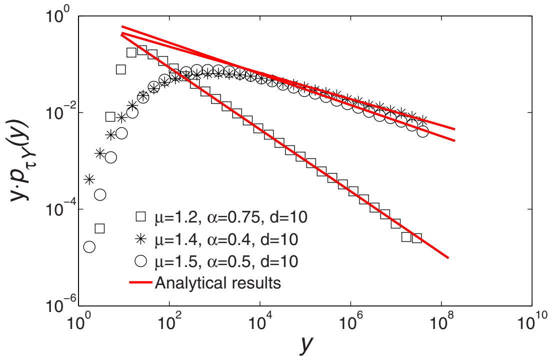

FIG. 4. (Color online) FPT PDFs $p_{\tau_Y}(y)$ of the anomalous diffusion $Y(t)=X(S_t)$, where $X(\tau)$ is the two-sided skewed μ-stable Lévy motion and S_t is the inverse α-stable subordinator, with different values of α, μ, and d. Plotted is $yp_{\tau_Y}(y)$ vs y on a log-log scale. The red (solid) lines correspond to the slope "$-\alpha/\mu$" [Eq. (23)]. In all three cases $\Delta t=0.01$ and 10^6 realizations.

$$p_{\tau_Y}(y) \propto y^{-(\alpha/\mu+1)} \tag{23}$$

as $y\to\infty$. Comparing this result with Eqs. (14) and (19), we see that only in the case when the parent process $X(\tau)$ is two-sided skewed, the decay of $p_{\tau_Y}(y)$ depends on both parameters α and μ. The above analytical results are confirmed by the performed numerical analysis of the FPT problem (see Fig. 4).

TABLE I. Summary of the FPT PDFs of three cases of Lévy stable motions, where flights (μ) and subdiffusion (α) coexist: symmetric, one-sided, and two-sided skewed (the extreme case), defined by the parameter β. In all cases heavy tailed waiting times ($0<\alpha<1$) are assumed.

Symmetric	One-sided	Two-sided skewed (extreme case)
$\beta=0,\ 0<\mu<2$ $y^{-(\alpha/2+1)}$	$\beta=1,\ 0<\mu<1$ $y^{-(\alpha+1)}$	$\beta=-1,\ 1<\mu<2$ $y^{-(\alpha/\mu+1)}$

V. CONCLUSIONS

We have investigated the FPT in a one-dimensional system displaying a competition between subdiffusion and Lévy flights under the influence of a constant potential (see Table I). We have solved the FPT problem analytically using the subordination formulation and confirmed the results numerically by using the recently developed algorithm for simulating trajectories of anomalous diffusion. We have explored the FPT problem for three subclasses of Lévy stable motions and have shown that the FPT PDFs in all the considered cases are heavy tailed.

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